Solecki submeasures and densities on groups

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Densities and submeasures on sets

Definition

A function $\mu : \mathcal{P}(X) \to [0,1]$ on a power-set of a set X is called:

- *monotone* if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of X;
- *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a *density* on X if μ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
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Haar measure of compact topological groups

Theorem (Haar, 1933)

Each compact topological group possesses a unique invariant probability σ-additive regular Borel measure $\lambda : \mathcal{B}(G) \to [0,1]$ defined on the σ-algebra of Borel subsets of G.

The uniqueness of λ implies that it is inversely and autoinvariant.

Problem

What about discrete groups? Do they have any canonical (sub)measures?

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Theorem (Banach, 1923)

There exists an invariant measure on the group of integers $\mathbb{Z}$.

Definition (von Neuman, 1929; Day, 1949)

A group G is called amenable if it admits a left-invariant measure $\mu : \mathcal{P}(G) \to [0,1]$.

Fact (Classics)

- *Each abelian group is amenable;*
- *A non-commutative free group is not amenable.*

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Invariant submeasures?

Conclusion: There are groups admitting no invariant measure :(

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What about invariant submeasures?
Do they always exist on any group?

Yes! $\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases}$ But this is trivial :(

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Are there any canonical non-trivial and useful invariant submeasure on a group?

Yes!! $\ddot\smile$

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Are there any canonical non-trivial and useful invariant submeasure on a group?

Yes!! $\ddot\smile$

Solecki submeasure on a group

Each group G possesses a canonical invariant submeasure $\sigma : \mathcal{P}(G) \to [0,1]$ defined by

$$\boxed{\sigma(A) = \inf_{F \in [G]^{<\omega}} \max_{x,y \in G} \frac{|F \cap xAy|}{|F|}.}$$

This submeasure is inversely and auto invariant.

The submeasure σ was thoroughly studied by Solecki and because of that we decided to name it the Solecki submeasure.

Example

The subset $A = 2\mathbb{Z}$ in $\mathbb{Z}$ has Solecki submeasure $\sigma(A) = \frac{1}{2}$.

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An alternative definition of the Solecki submeasure

The Solecki submeasure can be alternatively defined using finitely supported measures on G instead of finite subsets of G.

A measure μ on a set X is finitely supported if $\mu(F) = 1$ for some finite subset F. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures.

By $P(X)$ we denote the set of all measures on a set X and by $P_\omega(X)$ its subset consisting of finitely supported measures on X.

Theorem (Solecki, 2005)

Any subset A of a group G has Solecki submeasure

$$\sigma(A) = \inf_{\mu \in P_\omega(G)} \sup_{x,y \in G} \mu(xAy).$$

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The subadditivity of the Solecki submeasure

Given any subsets $A, B \subset G$ we need to prove that

$$\sigma(A \cup B) \le \sigma(A) + \sigma(B) + 2\varepsilon$$

for every $\varepsilon > 0$. Using the equivalent definition of the Solecki submesures, find two finitely supported probability measures $\mu_A, \mu_B \in P_\omega(G)$ such that

$$\max_{x,y \in G} \mu_A(xAy) < \sigma(A) + \varepsilon \quad \text{and} \quad \max_{x,y \in G} \mu_B(xBy) < \sigma(B) + \varepsilon.$$

Write $\mu_A = \sum_i \alpha_i \delta_{a_i}$ and $\mu_B = \sum_j \beta_j \delta_{b_j}$ and consider the convolution measure

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Observe that for any $x, y \in G$

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Consequently,

$$\mu(x(A \cup B)y) \le \mu(xAy) + \mu(xBy) < \sigma(A) + \sigma(B) + 2\varepsilon$$

and

$$\sigma(A \cup B) \le \sup_{x,y \in G} \mu(x(A \cup B)y)) \le \sigma(A) + \sigma(B) + 2\varepsilon.$$

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$$\sigma(A \cup B) \leq \sup_{x,y \in G} \mu(x(A \cup B)y)) \leq \sigma(A) + \sigma(B) + 2\varepsilon.$$

Observe that for any $x, y \in G$

$$\mu(xAy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xAy) = \sum_j \beta_j \sum_i \alpha_i \delta_{a_i}(xAyb_j^{-1}) =$$

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Left and right Solecki densities

The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

$$\sigma^L(A) = \inf_{F \in [G]^{<\omega}} \max_{x \in G} \frac{|F \cap xA|}{|F|} \qquad \sigma^R(A) = \inf_{F \in [G]^{<\omega}} \max_{y \in G} \frac{|F \cap Ay|}{|F|}$$

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It is clear that $\sigma_L \le \sigma^L \le \sigma \ge \sigma^R \ge \sigma_R$.
If the group G is abelian, then $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.
The densities $\sigma_L, \sigma^L, \sigma_R, \sigma^R$ are (auto) invariant but not inversely invariant in general. However

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Solecki submeasures versus Solecki densities

A group G is called an FC-group if each $x \in G$ has finite conjugacy class $x^G = \{gxg^{-1} : g \in G\}$.

abelian group $\Rightarrow$ FC-group $\Rightarrow$ amenable group

Theorem (Solecki, 2005)

1. *A group G is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.*
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In the free group $F_2 = \langle a, b\rangle$ consider the set A of irreducible words that start with a or a^{-1}.

The set A has right Solecki density $\sigma^R(A) = 0$ since for every set $F = \{b, b^2, \dots, b^n\}$, $n \in \mathbb{N}$, we get $\sup_{y\in G} |F \cap Ay| \le 1$ which implies $\sigma^R(A) \le \sup_{y\in G} \frac{|F\cap Ay|}{|F|} \le \frac{1}{n}$. By analogy we can prove that $\sigma^R(A) = 0$.

Then $\sigma^L(A^{-1}) = \sigma^R(A) = 0$ and $\sigma^L(B^{-1}) = \sigma^R(B) = 0$ and

$$F_2 = (A \cap A^{-1}) \cup (A \cap B^{-1}) \cup (B \cap A^{-1}) \cup (B \cap B^{-1})$$

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A minimax characterization of Solecki densities

The Kelley intersection number $I(\mathcal{F})$ of a family $\mathcal{F}$ of subsets of a set X is defined as

$$I(\mathcal{F}) = \inf_{F_1,\dots,F_n\in\mathcal{F}} \sup_{x\in X} \frac{1}{n}\sum_{i=1}^{n} \chi_{F_i}(x).$$

Theorem (B., 2012)

For a subset A of a group G we get

$$\inf_{\mu\in P_\omega(G)} \sup_{y\in G} \mu(Ay) = \sigma_R(A) = I(\{xA\}_{x\in G}) = \sup_{\mu\in P(G)} \inf_{x\in G} \mu(xA).$$

Here $P(G)$ stands for the set of all (finitely additive probability) measures on X.

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The upper Banach density on an amenable group

The upper Banach density $d^*(A)$ of a subset A of an amenable group G is defined as

$$d^*(A) = \sup_{\mu \in P_l(G)} \mu(A)$$

where $P_l(G)$ denotes the set of all left-invariant measures on X.

It is clear that the upper Banach density $d^* : \mathcal{P}(G) \to [0,1]$ is a left-invariant submeasure on each amenable group G.
The Minimax Theorem describing the right Solecki density implies:

Corollary (B., 2013)

*For any amenable group G we get $\sigma_R = \sigma^R = d^*$.
Consequently the right Solecki density on G is subadditive.*

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Solecki-amenable groups

Definition

A group G is Solecki-amenable if its Solecki density σ_R is subadditive.

Amenable group $\Rightarrow$ Solecki-amenable

Problem (Solecki, 2005)

Is each Solecki-amenable group amenable?

Theorem (B., 2012)

For a group G the following conditions are equivalent:

1. *G is amenable;*
2. *$G \times \mathbb{Z}$ is Solecki-amenable;*
3. *for every $n \in \mathbb{N}$ there is a finite group F of cardinality $|F| \geq n$ such that the product $G \times F$ is a Solecki-amenable group;*
4. *$\sigma_R(f) + \sigma_R(1 - f) \geq 1$ for any fuzzy set $f : G \to [0, 1]$.*

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Solecki one, null, and positive sets

A subset A of a group G is called

- Solecki null if $\sigma(A) = 0$;
- Solecki positive if $\sigma(A) > 0$;
- Solecki one if $\sigma(A) = 1$.

Solecki one sets can be characterized as follows:

Proposition

A subset A of a group G is Solecki one if and only if for each finite subset $F \subset G$ there are points $x, y \in G$ such that $xFy \subset A$.

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A subset A of a group G is Solecki one if and only if for each finite subset $F \subset G$ there are points $x, y \in G$ such that $xFy \subset A$.

Solecki one, null, and positive sets

A subset A of a group G is called

- Solecki null if $\sigma(A) = 0$;
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The ideal of Solecki null sets

The subadditivity of the Solecki submeasure σ implies that the Solecki null sets of a group G form an invariant ideal $\mathcal{S}_G$ on G.

Problem

Given a group G, study the properties of the ideal $\mathcal{S}_G$.
In particular, calculate its cardinal characteristics

$$\mathrm{add}(\mathcal{S}_G) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{S}_G,\ \cup\mathcal{A} \notin \mathcal{S}_G\},$$
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For each infinite group G we get

$$\begin{array}{ccccccc} & & \mathrm{non}(\mathcal{S}_G) & \longrightarrow & \mathrm{cof}(\mathcal{S}_G) & \longrightarrow & 2^{|G|} \\ & & \uparrow & & \uparrow & & \uparrow \\ \omega & \longrightarrow & \mathrm{add}(\mathcal{S}_G) & \longrightarrow & \mathrm{cov}(\mathcal{S}_G) & \longrightarrow & |G| \end{array}$$

Example (Not exciting)

For any infinite countable group G

$$\omega = \mathrm{add}(\mathcal{S}_G) = \mathrm{non}(\mathcal{S}_G) = \mathrm{cov}(\mathcal{S}_G) < \mathrm{cof}(\mathcal{S}_G).$$

If G is abelian, then $\omega = \mathrm{add}(\mathcal{S}_G) = \mathrm{cov}(\mathcal{S}_G)$ and $\mathrm{non}(\mathcal{S}_G) = |G|$.

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The exciting example

In the group $G = FS_\kappa$ of finitely supported permutations of the cardinal κ consider the countable subgroup $H = FS_\omega$ consisting of all permutations $f : \kappa \to \kappa$ with finite support

$$\mathrm{supp}(f) = \{x \in \kappa : f(x) \neq x\} \subset \omega.$$

It can be shown that $\sigma(H) = 1$.
So, $H \notin \mathcal{S}_G$ and

$$\omega \le \mathrm{add}(\mathcal{S}_G) \le \mathrm{non}(\mathcal{S}_G) \le |H| = \omega.$$

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Calculate $\mathrm{cov}(\mathcal{S}_G)$ for the group FS_κ.

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Solecki null sets in compact topological groups

Theorem

If a group G admits a homomorphism onto an infinite compact Hausdorff group, then $\mathrm{non}(\mathcal{S}_G) \ge \mathrm{cov}(\mathcal{E})$.

Here $\mathrm{cov}(\mathcal{E})$ denotes the smallest cardinality of a cover of an infinite compact metrizable group by closed Haar null subsets. This cardinal was thoroughly studied by Bartoszynski and Shelah.

Corollary

For any infinite cardinal κ the group FS_κ admits no homomorphism onto an infinite compact Hausdorff topological group.

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The Solecki submeasure σ versus the Haar measure λ

Let G be a compact topological group and λ be its Haar measure.
For a subset $A \subset G$ let $\bar{A}$ be the closure of A in X and
$A^\bullet$ (resp. A°) be the largest open set $U \subset G$ such that $U \setminus A$ is meager in G (resp. empty).
It is clear that A° is the interior of A and $A^\circ \subset A^\bullet \subset \bar{A}$.
Example: Each dense G_δ-set $A \subset G$ has $A^\bullet = G$.

Theorem

Any subset A of a compact topological group G has

$$\max\{\lambda_*(A), \lambda(A^\bullet)\} \leq \sigma(A) \leq \lambda(\bar{A}).$$

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Each closed subset A of a compact topological group G has $\sigma(A) = \lambda(A)$.

This means that the Haar measure λ is completely determined by the Solecki submeasure:

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For a compact Hausdorff topological group G its Haar measure is a unique regular σ-additive Borel measure λ such that $\lambda(A) = \sigma(A)$ for each closed subset $A \subset G$.

So, the Haar measure, being a topologo-algebraic object has more essential algebraic component than could be expected.

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Ramsey Applications of the Solecki submeasure

Van der Waerden and Gallai's Theorem

Theorem (Van der Waerden, 1927)

For any partition $\mathbb{Z} = A_1 \cup \dots \cup A_n$ of integers there is a cells A_i of the partition containing arbitrarily long arithmetic progressions.

This theorem can be deduced from a more general:

Theorem (Gallai, $\le$ 1933)

For any finite partition $G = A_1 \cup \dots \cup A_n$ of the group $G = \mathbb{Z}^n$ there is a cell A_i of the partition containing the homothetic copy of each finite set $F \subset G$.

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By a homothetic copy of a set F in a group G we understand the image $h(F)$ of F under a polynomial map $h : G \to G$ of the form $h(x) = a_0 x a_1 \dots a_{n-1} x a_n$ for some constants $c_0, \dots, c_n \in G$.

If $n = 1$, then $h(x) = c_0 x c_1$ and we say that $h(F) = c_0 F c_1$ is a translation copy of the set F.

Homothetic properties of subsets in groups

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Generalizing Van der Waerden and Gallai Theorem

Theorem (B., 2012)

If a subset A of a group G is:

- *Solecki one, then A contains a translation copy of each finite subset $F \subset G$;*
- *Solecki positive, then A contains a homothetic copy of each finite subset $F \subset G$.*

This theorem combined with the subadditivity of the Solecki submeasure implies the following generalization of Gallai's Theorem:

Corollary

For any finite partition $G = A_1 \cup \cdots \cup A_n$ of any group G there is a cell A_i of the partition containing a homothetic copy of each finite subset $F \subset G$.

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Green-Tao Theorem

Theorem (Green, Tao, 2008)

The set of prime numbers contains arbitarily long arithmetic progressions.

Unfortunately, this theorem cannot be deduced from our result because of:

Proposition

The set of primes is Solecki null in the group $\mathbb{Z}$.

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Theorem (Steinhaus-Weil)

For any measurable subset A of positive Haar measure $\lambda(A)$ in a compact topological group G the difference set AA^{-1} is a neighboorhood of zero in G.

Problem

Can the Haar measure in this theorem replaced with the Solecki submeasure σ or the right Solecki density σ^R?

Answer

Partially Yes! (for the right Solecki density σ^R).

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A subset A of a group G is right-Solecki one iff for each finite subset $F \subset G$ there is a point $y \in G$ such that $Fy \subset A$.

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Right Solecki density and packing index

For a subset A of a group G the cardinal

- $\mathrm{pack}_L(A) = \sup\{|E| : E \subset G \ \ (xA)_{x \in E}$ is disjoint$\}$ is called the left packing index of A;
- $\mathrm{cov}_L(A) = \min\{|E| : E \subset G, \ EA = G\}$ is called the left covering number of A.

Theorem

$$\mathrm{cov}_L(AA^{-1}) \leq \mathrm{pack}_L(A) \leq \frac{1}{\sigma^R(A)}.$$

Corollary

If an (analytic) subset A a Polish group G is right-Solecki positive, then AA^{-1} is not meager (and $AA^{-1}AA^{-1}$ is a neighborhood of the unit) in G.

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Theorem (Protasov-B., $\le$ 2003)

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For any partition $G = A_1 \cup \cdots \cup A_n$ of a group G there is $i \le n$ such that $\mathrm{cov}_L(A_i A_i^{-1}) \le 2^{2^{n-1}-1}$.

A partial answer to Protasov's Problem

A subset $A \subset G$ is called *inner-invariant* if $\forall x \in G\ \ xAx^{-1} = A$.

Theorem (B.-Protasov-Slobodianiuk, 2013)

Let $G = A_1 \cup \dots \cup A_n$ be a partition of a group G. If G is Solecki-amenable or all sets A_i are inner-invariant, then $\mathrm{cov}_L(A_iA_i^{-1}) \le n$ for some i.

Proof.

If G is Solecki-amenable, then the right Solecki submeasure σ_R is subadditive and then $\sigma_R(A_i) \ge 1/n$ for some i and hence

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Equivalently, it can be defined as the smallest topology on G in which every homomorphism $h : G \to K$ to a compact Hausdorff topological group K is continuous.
It this case we can assume that $K = \prod_{n=1}^{\infty} O(n)$.

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Difference sets and Bohr open sets

Now we shall generalize results of Bogoliuboff, Følner, Cotlar, Ricabarra (1954), Ellis, Keynes (1972), Beiglböck, Bergelson, Fish (2010).

Theorem (B., 2013)

For each right-Solecki positive set A in an amenable group G there are a Bohr open neighborhood $U \subset G$ of the unit 1_G and a right-Solecki null subset $N \subset G$ such that $U \setminus N \subset AA^{-1}$.

Corollary (B., 2013)

For any right-Solecki positive set A, B in an amenable group G the set $B^{-1}AA^{-1}$ has non-empty interior and $AA^{-1}BB^{-1}$ is a neighborhood of the unit 1_G in the Bohr topology on the group G.

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The Bohr topology on a group G is trivial if and only if each homomorphism $h : G \to K$ to a compact Hausdorff topological group K is constant.

Examples of groups with trivial Bohr topology are:

- the group S_X of all permutations of an infinite set X;
- the group A_X of all even finitely supported permutations of an infinite set X.

The group A_X is locally finite and hence amenable.

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Theorem

An amenable group G has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell A_i with $A_iA_i^{-1}A_i = G$.

A group G is odd if every element of G has odd order.

Theorem (B.-Nykyforchyn-Gavrylkiv, 2008)

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Some corollaries for permutation groups

Corollary

If a subset A of an infinite alternating group $G = A_X$ is right-Solecki positive, then $AA^{-1}A = G$.

Problem

Is $AA^{-1}A = G$ for each (inner-invariant) right-Solecki positive set A in an infinite permutation group $G = S_X$?

Applying some results of Bergman (2006) it is possible to prove:

Theorem (B., 2013)

For any inner-invariant Solecki positive subset A of an infinite permutation group $G = S_X$ we get $(AA^{-1})^{18} = G$.

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References

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